EFFECTIVE FINITE TEMPERATURE PARTITION FUNCTION FOR FIELDS ON NON-COMMUTATIVE FLAT MANIFOLDS

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Abstract. The first quantum correction to the finite temperature partition function for a self-interacting massless scalar field on a $D-$dimensional flat manifold with p non-commutative extra dimensions is evaluated by means of dimensional regularization, suplemented with zeta-function techniques. It is found that the zeta function associated with the effective one-loop operator may be nonregular at the origin. The important issue of the determination of the regularized vacuum energy, namely the first quantum correction to the energy in such case is discussed.

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1. Introduction

Quantum field theories on non-commutative spaces generalize the familiar structure of conventional field theories. A motivation for considering these models is their appearance in $M-$theory and in the theory of strings [1, 2, 3, 4, 5, 6, 7], and also the fact that they are perfectly consistent formulations by themselves. It has been shown, for instance, that non-commutative gauge theories describe the low energy excitations of open strings on $D-$branes in a background Neveu-Schwarz two-form field B [1, 2, 3].

Recently, the non-commutative perturbative dynamics on $D-$ dimensional manifolds have been investigated [8] and the Kaluza-Klein spectrum for interacting scalars and vector fields has been calculated in [9]; it is consistent with formulas found in [8, 10, 11], where a connection between calculations performed in string theory and some others, done

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in field theory on a non-commutative torus, has been made. The one-loop Casimir energy of scalar and vector fields on a non-commutative space like $\mathbb{R}^{1,d}\otimes T^2_\theta$ (where $\mathbb{R}^{1,d}$ is a flat $(d+1)-$dimensional Minkowski space and T^2_θ a two-dimensional non-commutative torus) has been calculated in [12]. The higher dimensional non-commutative torus case has been investigated in [13, 14]; termal effects in perturbative non-commutative gauge theories have been actively studied in Refs. [15, 16, 17, 18, 19, 20].

In this paper we will calculate - using dimensional regularization implemented with zeta-function techniques - the partition function at inverse temperature β for massless scalar fields defined on $D-$dimensional manifolds with compact and non compact non-commutative dimensions. This can be considered as *the first* quantum correction to the total partition function. Since we will compute a functional determinat of a one-loop operator, our computation is, as well known, equivalent to summing an infinite number of one-loop diagrams [8]. The relevance of this contribution will be discussed later.

The action of the massless interacting scalar fields we are considering has the form

$$S_{(\mathrm{scalar})} = \int \left(\frac{1}{2}(\partial\,\phi)^2 + \frac{\lambda}{r!}\phi\star\phi\star\ldots\star\phi\right)d^Dx. \tag{1.1}$$

As is well known, in ultrastatic spacetimes finite temperature effects can be easily described with the help of the imaginary time formalism - in which the fields are assumed to be periodic with period β. For this reason, we will consider manifolds with topology $M = S^1\otimes\mathbb{R}^{d-1}\otimes X^p_\theta$, where $p = D-d$ and $\dim M = D$. In the non compact case $X^p_\theta = \mathbb{R}^p_\theta$, while in the compact case $X^p_\theta = T^p_\theta$ is a non-commutative torus. If we denote by x^μ the non-commutative coordinates, we assume that

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1.2}$$

with $\mu,\nu = 1,...p$, and $\theta^{\mu\nu} = \theta\sigma^{\mu\nu}$ is a real, non-singular, antisymmetric matrix with entries ± 1, being θ the non-commutative parameter. X^p_θ may be regarded as a non-commutative associative algebra, with elements given by ordinary continuous functions on X^p_θ whose product is given by the Moyal bracket or $(\star)-$product of functions:

$$(F\star G)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\nu}\right)F(x+\alpha)G(x+\beta)|_{\alpha=\beta=0}. \tag{1.3}$$

Since we have to deal with a renormalizable scalar field theory, the number r of scalar fields in the $(\star)-$product should be chosen as an integer, given by $r = 2D/(D-2)$. Clearly, the only possible choices are $D=3$, $r=6$, $D=4$, $r=4$, and $D=6$, $r=3$.

We are interested in the one-loop approximation for a theory with the action (1.1). It has been shown that in the massless case, the operator $\mathcal{L}$, related to the one-loop contribution, has the formal aspect [8]:

$$\mathcal{L} = k^2 + \frac{\Lambda}{k \circ k} + M^2, \tag{1.4}$$

where k^2 is the $D-$dimensional Laplacian on $\mathbb{R}^D$ and $k \circ k = \theta^2 k_\mu k_\mu$. The effective mass term M^2 depends on λ and corresponds to the planar diagram contributions, while the non local λ term represents the non-planar diagram contribution.

We are also assuming that the time coordinate is commutative (unitarity of the field theory) and we denote by $\tau = -it$ the compactified imaginary time with period β. Thus, the finite temperature one-loop operator is

$$\mathcal{L} = L_d + L_p + \Lambda(\theta^2 L_p)^{-1}, \tag{1.5}$$

where $L_d = -\partial_\tau^2 + L_{d-1} = -\partial_\tau^2 + k_j^2 + M^2$ and $L_p = k_\mu^2$ are the Laplacian-like operators on the spaces $S^1 \otimes \mathbb{R}^{d-1}$ and X_θ^p, respectively.

We observe that in the *massive* case, the one-loop effective operator is much more complicated. It reads

$$\mathcal{L} = k^2 + m^2 + \frac{2\Lambda m}{(k \circ k)^{3/2}} K_1(2m(k \circ k)^{-1/2}) + M^2. \tag{1.6}$$

This appears as an untractable form. Since $K_1(z) \to 1/z$ as $z \to 0$, in the massless limit, one obtains Eq. (1.4). If we take the commutative limit, namely $\theta \to 0$ in Eq. (1.6), one gets $\mathcal{L} = k^2 + m^2 + M^2$, which is the correct one-loop operator in the commutative case. In Ref. [8], an approximate expression valid for small m has been considered and a mixing between UV and IR divergences has been noticed. It should be noted that the massless limit and the limit $\theta \to 0$ do not commute, namely it is not possible to consider the $\theta \to 0$ in Eq. (1.4). In this paper, for the sake of simplicity, we shall consider the massless case only, bearing in mind possible limitations of this choice.

2. Regularization of the partition function

We shall make use of dimensional regularization in the proper time formalism, implemented by zeta-function techniques [21, 22, 23]. The logarithm of the one-loop effective partition function is regularized by means of (μ is a renormalization parameter)

$$\log Z_\beta(\varepsilon) = -\frac{1}{2}\left(\log\det \mathcal{L}/\mu^2\right)_\varepsilon = \frac{1}{2}\mu^{2\varepsilon}\int_0^\infty dt t^{\varepsilon-1}\mathrm{Tr} e^{-t\mathcal{L}}, \tag{2.1}$$

where ε is a small regularization parameter. Since we are making use of the Gaussian approximation, what we are going to compute is the first quantum correction to the partition function.

The object of interest is the heat kernel trace $\omega(t)$, defined for $t > 0$ by $\omega(t|\mathcal{L}) = \mathrm{Tr}\, e^{-t\mathcal{L}}$. The zeta function $\zeta(s|\mathcal{L})$ and $\omega(t|\mathcal{L})$ are related by the Mellin transform:

$$\zeta(s|\mathcal{L}) = \frac{1}{\Gamma(s)} \int_0^\infty \omega(t|\mathcal{L}) t^{s-1} dt, \quad \text{for } \Re s > D/2. \tag{2.2}$$

Then, one has

$$\log Z_\beta(\varepsilon) = \frac{1}{2} \mu^{2\varepsilon} \Gamma(\varepsilon) \zeta(\varepsilon|\mathcal{L}). \tag{2.3}$$

If the zeta function of the operator $\mathcal{L}$ is regular at the origin, the one-loop divergences and the finite part of the logarithm of the partition function can be expressed, respectively, in terms of the zeta function and its derivative evaluated at the origin. In this case the vacuum energy (Casimir energy) can be defined resorting to the usual thermodynamical relation

$$< E > \; = \; - \lim_{\beta \to \infty} \partial_\beta \mathrm{log} Z_\beta, \tag{2.4}$$

where the renormalized partition function is simply

$$\log Z_\beta = \frac{1}{2} \left(\frac{d}{ds} \zeta(s|\mathcal{L})|_{s=0} + \mathrm{log} \mu^2 \zeta(0|\mathcal{L}) \right). \tag{2.5}$$

For example, in the case $S^1 \otimes Y$, one gets (see Ref. [23])

$$< E > \; = \; \frac{1}{2} \mathrm{PP}\, \zeta(-\frac{1}{2}|\mathcal{L}_Y) + (1 + \mathrm{log} 2\mu) \mathrm{Res}\, \zeta(-\frac{1}{2}|\mathcal{L}_Y), \tag{2.6}$$

where PP denotes the principal part of the zeta function, given by

$$\mathrm{PP}\, \zeta(-\frac{1}{2}|\mathcal{L}_Y) = \lim_{s \to 0} \left(\zeta(s - \frac{1}{2}|\mathcal{L}_Y) + \frac{A_{D/2}(\mathcal{L}_Y)}{2\sqrt{\pi} s} \right), \tag{2.7}$$

with $\mathcal{L}_Y$ being the spatial operator acting on a manifold Y, namely $\mathcal{L} = -\partial_\tau^2 + \mathcal{L}_Y$, and $A_{D/2}$ the corresponding heat-kernel coefficient. This prescription has been introduced in [24] and rederived, by using zeta-function regularization, in [25, 26]. In the following we will investigate the analytic continuation of $\zeta(s|\mathcal{L})$.

Making use of heat-kernel techniques, one can show that the zeta function of the one-loop operator may, in general, be nonregular at the origin. Indeed, we have

$$\mathrm{Tr}\ e^{-t\mathcal{L}} = \mathrm{Tr}\ e^{-tL_d} \mathrm{Tr}\ e^{-t\mathcal{L}_X}, \tag{2.8}$$

where we have introduced the pseudo-differential operator $\mathcal{L}_X = L_p + \Lambda(\theta^2 L_p)^{-1}$. The first short t asymptotics is standard and reads (for flat manifolds)

$$\omega(t|L_d) \simeq \frac{\mathrm{Vol}(\mathbb{R}^d)}{(4\pi t)^{d/2}}, \quad \text{as } t \to 0^+. \tag{2.9}$$

The second one involves the operator $\mathcal{L}_X$ and contains logarithmic terms (see, for example, [27] and references quoted there). For the cases we are interested in, we have

$$\omega(t|\mathcal{L}_X) \simeq \sum_{r=0}^{\infty} A_r t^{r-p/2} + \sum_{k=0}^{\infty} B_k t^{2k+p/2} \log t. \tag{2.10}$$

Thus, the contribution containing logarithmic terms is

$$\omega(t|\mathcal{L}) \simeq \frac{\mathrm{Vol}(\mathbb{R}^d)}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} B_k t^{2k+(p-d)/2} \log t + \text{non log terms}. \tag{2.11}$$

Since a pole of the zeta function at the origin corresponds to having the pure log term $\log t$, it turns out that a pole will appear if and only if $d = p + 4k\,,\ k \in \mathbb{Z}_+$ and since p is even, D also has to be even. The appearance of a pole at the origin of the zeta function is quite unsual. It is in common with quantum fields defined on higher dimensional cones [28, 29] and on 4-dimensional spacetimes with a non compact (but of finite volume) spatial hyperbolic section [30].

One can calculate the effective partition function evaluating a regularized functional determinant of $\zeta(s|\mathcal{L})$, Eq. (2.1), which can be rewritten as

$$\log Z_\beta(\varepsilon) = \frac{1}{2}\mu^{2\varepsilon}\Gamma(\varepsilon)\zeta(\varepsilon|\mathcal{L}). \tag{2.12}$$

Recall we are going to compute a functional determinant, and thus we are effectively adding up an infinite number of one-loop diagrams, planar and non-planar ones. Our calculation should be viewed as an attempt at an implementation of the background field method to the non-commutative case.

In order to go on, we need the evaluation of the analytic continuation of the zeta function $\zeta(s|\mathcal{L})$. Starting from the Mellin transform and making use of the Poisson-Jacobi resummation formula, a standard calculation leads to

$$\zeta(s|\mathcal{L}) = \frac{\beta}{(4\pi)^{\frac{1}{2}}\Gamma(s)} \Bigg(\Gamma(s-\frac{1}{2})\zeta(s-\frac{1}{2}|\mathcal{L}_Y) + 2\sum_{n=1}^{\infty} \int_0^{\infty} dt\, t^{s-\frac{3}{2}} e^{-\frac{n^2\beta^2}{4t}} \omega(t|\mathcal{L}_Y) \Bigg). \tag{2.13}$$

Here $\mathcal{L}_Y = L_{d-1} + \mathcal{L}_X$. In order to evaluate the high temperature expansion the second term on the r.h.s. of Eq. (2.13) can be conveniently rewritten as a Mellin-Barnes integral (see [23]). As a result, we have

$$\zeta(s|\mathcal{L}) = \frac{\beta}{(4\pi)^{\frac{1}{2}}\Gamma(s)} \left(\Gamma(s-\frac{1}{2})\zeta(s-\frac{1}{2}|\mathcal{L}_Y) \right.$$
$$\left. + \frac{1}{2\pi i}\int_{\Re z=\frac{D}{2}} dz \left(\frac{\beta}{2}\right)^{-z} \Gamma(\frac{z}{2})\zeta_R(z)\Gamma(s+\frac{z-1}{2})\zeta(s+\frac{z-1}{2}|\mathcal{L}_Y) \right), \tag{2.14}$$

where $\zeta_R(z)$ is the Riemann zeta function. In the equations above, the first part represents the vacuum contribution, while the second one is the statistical sum contribution. It is also clear that one needs to know the meromorphic structure of the zeta function $\zeta(s|\mathcal{L}_Y)$, which can be obtained from $\zeta(s|\mathcal{L}_X)$. Making an expansion in terms of the effective mass M^2, a direct computation gives

$$\zeta(z|\mathcal{L}_Y) = \frac{\mathrm{Vol}(\mathbb{R}^{d-1})}{(4\pi)^{\frac{d-1}{2}}} \sum_{k=0}^{\infty} (-M^2)^k \frac{\Gamma(z+k-\frac{d-1}{2})}{k!\Gamma(z)} \zeta(z+k-\frac{d-1}{2}|\mathcal{L}_X). \tag{2.15}$$

3. The spectral zeta function and the regularized vacuum energy

First of all, let us consider the simple case of a non-compact, non-commutative manifold $X^p_\theta = \mathbb{R}^p_\theta$ (i.e., the non-commutative Euclidean space). The heat kernel trace reads

$$\omega(t|\mathcal{L}_X) = \frac{2\mathrm{Vol}(\mathbb{R}^p)}{(4\pi)^{p/2}\Gamma(p/2)} a^{p/2} K_{p/2}(2at), \tag{3.1}$$

where $K_\nu(z)$ is the Mac Donald function and $a^2 = \Lambda\theta^{-2}$. It is easy to show that the short$-t$ asymptotics of $\omega(t|\mathcal{L}_X)$ is of the kind given by Eq. (2.10). The Mellin transform leads to

$$\zeta(z|\mathcal{L}_X) = \frac{\mathrm{Vol}(\mathbb{R}^p)}{2(4\pi)^{p/2}} \frac{\Gamma(\frac{2z+p}{4})\Gamma(\frac{2z-p}{4})}{\Gamma(p/2)\Gamma(z)} a^{p/2-z}. \tag{3.2}$$

Eq. (3.2) gives the analytic continuation of the zeta function.

In the case of a compact non-commutative manifold (i.e., the non-commutative tori) $X^p_\theta = T^p_\theta$, for large $\Re\, s$, the spectral zeta function associated with the operator $\mathcal{L}_X$ reads [19, 12, 14]:

$$\zeta(s|\mathcal{L}_X) = \sum_{\mathbf{n}\in\mathbb{Z}^p/\{\mathbf{0}\}} \varphi(\mathbf{n})^{-s} \left[1 + \Lambda\theta^{-2}R^4\varphi(\mathbf{n})^{-2}\right]^{-s}, \tag{3.3}$$

where R is the compactification radius. The analytic continuation can be achieved by using binomial expansion ([31, 21, 22, 23]). As a result, for $C = a^2R^4 < 1$ one has

$$\zeta(s|\mathcal{L}_X) = R^{2s} \sum_{\ell=0}^{\infty} \frac{(-C)^\ell \Gamma(s+\ell)}{\ell! \Gamma(s)} Z_p \left| \begin{matrix} \mathbf{0} \\ \mathbf{0} \end{matrix} \right| (2s+4\ell)\,, \tag{3.4}$$

where $Z_p \left|{}^{\mathbf{g}}_{\mathbf{h}}\right| (s,\varphi)$ is the p-dimensional Epstein zeta function associated with the quadratic form $\varphi[a(\mathbf{n}+\mathbf{g})] = \sum_j a_j(n_j+\mathrm{g}_j)^2$. For $\Re\, s > p$, $Z_p \left|{}^{\mathbf{g}}_{\mathbf{h}}\right| (s,\varphi)$ is given by the formula

$$Z_p \left| \begin{matrix} \mathrm{g}_1 \dots \mathrm{g}_p \\ h_1 \dots h_p \end{matrix} \right| (s,\varphi) = \sum_{\mathbf{n}\in\mathbf{Z}^p}{}' \left(\varphi[a(\mathbf{n}+\mathbf{g})]\right)^{-s/2} \exp\left[2\pi i(\mathbf{n},\mathbf{h})\right], \tag{3.5}$$

where g_j and h_j are some real numbers [32], and the prime means omitting the term with $(n_1, n_2, ..., n_p) = (\mathrm{g}_1, \mathrm{g}_2, ..., \mathrm{g}_p)$ if all of the g_j are integers. The functional equation for $Z_p \left|{}^{\mathbf{g}}_{\mathbf{h}}\right| (s,\varphi)$ reads

$$Z_p \left| \begin{matrix} \mathbf{g} \\ \mathbf{h} \end{matrix} \right| (s,\varphi) = (\det a)^{-1/2} \pi^{\frac{1}{2}(2s-p)} \frac{\Gamma(\frac{p-s}{2})}{\Gamma(\frac{s}{2})} \exp[-2\pi i(\mathbf{g},\mathbf{h})]$$

$$\times Z_p \left| \begin{matrix} \mathbf{h} \\ -\mathbf{g} \end{matrix} \right| (p-s,\varphi^*), \tag{3.6}$$

where $\varphi^*[a(\mathbf{n}+\mathbf{g})] = \sum_j a_j^{-1}(n_j+\mathrm{g}_j)^2$. Formulas (3.4) and (3.6) give the analytic continuation of the zeta function.

Alternatively, a powerful expression for the analytic continuation, that extends (to arbitrary number of dimensions and to more general Epstein-like zeta functions) the Chowla-Selberg formula for the homogeneous Epstein zeta function in two-dimensions, has been given recently [33, 34, 35]. It provides an expression valid in the whole of the complex plane, exhibiting the poles explicitly, and under the form of a power-fast convergent series. For the compact case, Eq. (3.3), the formula reads:

$$\zeta(s|\mathcal{L}_Y) = \frac{2^{s-d+2}\mathrm{Vol}(\mathbb{R}^{d-1})}{(2\pi)^{(d-1)/2}\Gamma(s)} \sum_{\ell=0}^{\infty}\sum_{j=0}^{p-1} \frac{(-4\Lambda\theta^{-2})^\ell \Gamma(s+\ell-\frac{d-1}{2})}{\ell!\, \Gamma(s+2\ell-\frac{d-1}{2})\, (\det \mathbf{A}_j)^{1/2}}$$

$$\times \Bigg[\pi^{j/2} a_{p-j}^{-s-2\ell+(d+j-1)/2} \Gamma(s+2\ell-\frac{d+j-1}{2}) \zeta_R(2s+4\ell-d-j+1)$$

$$+4\pi^{s+2\ell-(d-1)/2} a_{p-j}^{-s/2-\ell-(d+j-1)/4} \sum_{n=1}^{\infty} \sum_{\mathbf{m}_j\in\mathbb{Z}^j}{}' n^{(d+j-1)/2-s-2\ell}$$

$$\times \left(\mathbf{m}_j^t \mathbf{A}_j^{-1} \mathbf{m}_j\right)^{s/2+\ell-(d+j-1)/4} K_{(d+j-1)/2-s-2\ell}\left(2\pi n\sqrt{a_{p-j}\mathbf{m}_j^t \mathbf{A}_j^{-1} \mathbf{m}_j}\right)\Bigg], \tag{3.7}$$

where $a_j = R_j^{-2}$, R_j being the compactification radius corresponding to the jth coordinate (above, and in what follows, only the particular case $R_j = R, \forall j$, is considered). Note that the term $j = 0$ in the second sum reduces to: $a_p^{s-2\ell}\Gamma(s+2\ell)\zeta_R(2s+2\ell)$, which coincides with the result of the 1-dimensional case ($p = 1$), as it should. Here $\mathbf{A}_j$ is the submatrix of $\mathbf{A} = \mathrm{diag}\,(a_1, \dots, a_p)$ made up of the last j rows and columns: $\mathbf{A}_j = \mathrm{diag}\ (a_{p-j+1}, \dots, a_p)$.

Now we have in our hands all the ingredients necessary to study the limit ε goes to 0 in Eq. (2.3), making use of Eqs. (3.4) and (3.6). The vacuum contribution to the one-loop effective action has been investigated recently in [14]. The presence of poles at the origin in the zeta function of Eq. (3.4) (see also Ref. [14]) imposes a reanalysis of the renormalization. This can proceed as follows.

For small ε one has

$$\log Z_\beta(\varepsilon) = \frac{1}{2}\mu^{2\varepsilon}\Gamma(\varepsilon)\zeta(\varepsilon|\mathcal{L}) = \frac{W_0}{\varepsilon^2} + \frac{W_1}{\varepsilon} + W_2 + \mathcal{O}(\varepsilon). \tag{3.8}$$

It is convenient to introduce the "regular" zeta function

$$\xi(s|\mathcal{L}) = s\zeta(s|\mathcal{L}). \tag{3.9}$$

Multiplying both sides of Eq. (3.8) by ε^2, we obtain the following expressions for the coefficients W_j:

$$W_0 = \frac{1}{2}\xi(0|\mathcal{L}) = \frac{1}{2}\mathrm{Res}\ \left(\zeta(\varepsilon|\mathcal{L})|_{\varepsilon=0}\right)\ , \tag{3.10}$$

$$\begin{aligned} W_1 &= \frac{1}{2}\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\left(\mu^{2\varepsilon}\Gamma(1+\varepsilon)\xi(\varepsilon|\mathcal{L})\right) \\ &= \frac{1}{2}\left(\frac{d}{ds}\xi(s|\mathcal{L})|_{s=0} + (\log\mu^2 - \gamma)\xi(0|\mathcal{L})\right), \end{aligned} \tag{3.11}$$

$$\begin{aligned} W_2 &= \frac{1}{4}\lim_{\varepsilon\to 0}\frac{d^2}{d^2\varepsilon}\left(\mu^{2\varepsilon}\Gamma(1+\varepsilon)\xi(\varepsilon|\mathcal{L})\right) \\ &= \frac{1}{4}\left[\frac{d^2}{ds^2}\xi(s|\mathcal{L})|_{s=0} + 2(\log\mu^2-\gamma)\frac{d}{ds}\xi(s|\mathcal{L})|_{s=0}\right] \\ &+\frac{\xi(0|\mathcal{L})}{4}\left[(\log\mu^2-\gamma)^2 + \frac{d}{ds}\Psi(1+s)|_{s=0}\right], \end{aligned} \tag{3.12}$$

where $\Psi(s)$ is the logarithmic derivative of the Gamma function and $\gamma = -\Psi(1)$ is the Euler-Mascheroni constant. The coefficients W_0 and W_1 are the counterterms, while W_2 is the renormalized partition function. If the zeta function $\zeta(0|\mathcal{L})$ is regular at the origin, the quantity

defined by Eq. (3.12) reduces to the renormalized partition function, because

$$\xi(0|\mathcal{L}) = 0\,, \quad \frac{d}{ds}\xi(s|\mathcal{L})|_{s=0} = \zeta(0|\mathcal{L}),$$
$$\frac{d^2}{ds^2}\xi(s|\mathcal{L})|_{s=0} = \frac{d}{ds}\zeta(s|\mathcal{L})|_{s=0}. \tag{3.13}$$

3.1. **The regularized vacuum energy.** In order to evaluate the regularized vacuum energy, one needs the meromorphic structure of the zeta functions $\zeta(z|\mathcal{L})$ and $\zeta(z|\mathcal{L}_X)$. The latter can be obtained by making use of the short heat kernel expansion (2.9). A direct calculation gives

$$\Gamma(z)\zeta(z|\mathcal{L}_X) = \sum_{r=0}^{\infty}\frac{A_r}{z+r-p/2} - \sum_{k=0}^{\infty}\frac{B_k}{(z+2k+p/2)^2} + J(z), \tag{3.14}$$

where $J(z)$ is the analytic part. To be noted is the presence of second order poles associated with the logarithmic terms. Thus,

$$\Gamma(z)\zeta(z|\mathcal{L}_Y) = \sum_{k=0}^{\infty}\frac{(-M^2)^k}{k!}\left(\sum_{r=0}^{\infty}\frac{A_r(\mathcal{L}_Y)}{z+k+r-\frac{D-1}{2}}\right.$$
$$\left. - \sum_{k=0}^{\infty}\frac{B_k(\mathcal{L}_Y)}{(z+2k+\frac{p-d+1}{2})^2} + J(z|\mathcal{L}_Y)\right), \tag{3.15}$$

where we have introduced the Seeley-De Witt coefficients

$$A_r(\mathcal{L}_Y) = \frac{\mathrm{Vol}(\mathbb{R}^{d-1})}{(4\pi)^{\frac{d-1}{2}}}A_r, \qquad B_r(\mathcal{L}_Y) = \frac{\mathrm{Vol}(\mathbb{R}^{d-1})}{(4\pi)^{\frac{d-1}{2}}}B_r. \tag{3.16}$$

If D is even and if $\zeta(s|\mathcal{L})$ has a pole at the origin, Eqs. (2.4), (2.14), (3.12) and (3.15) lead to

$$< E > \;=\; \frac{1}{2}\mathrm{PP}\,\zeta(-\frac{1}{2}|\mathcal{L}_Y) + (1+\log 2\mu - \frac{\gamma}{2})\mathrm{Res}\,\zeta(-\frac{1}{2}|\mathcal{L}_Y)$$
$$+\frac{1}{4}B_{(D-2p)/4}(\mathcal{L}_Y)\left[\log^2\mu^2 - \left(\frac{\pi^2}{6} + (2-2\log 2-\gamma)^2\right)\right], \tag{3.17}$$

where now

$$\mathrm{PP}\zeta(-\frac{1}{2}|\mathcal{L}_Y) = \lim_{s\to 0}\left\{\zeta(s-\frac{1}{2}|\mathcal{L}_Y)\right.$$
$$\left. -\frac{1}{2\sqrt{\pi}}\left[\frac{B_{(D-2p)/4}(\mathcal{L}_Y)}{s^2} - \frac{1}{s}[A_{D/2}(\mathcal{L}_Y) + \Psi(-\frac{1}{2})B_{(D-2p)/4}(\mathcal{L}_Y)]\right]\right\}. \tag{3.18}$$

The coefficients $B_{(D-2p)/4}(\mathcal{L}_Y)$ are related to the residue of $\zeta(s|\mathcal{L})$ at the origin. This is the *new* prescription for the evaluation of the regularized

vacuum energy. When there is no pole at the origin for $\zeta(s|\mathcal{L})$, the prescription reduces to the standard one given by (2.7).

3.2. **Non-Commutative Euclidean space.** In our situation, i.e. in the case of massless scalar field defined on a D manifold with noncommuting coordinates, the relevant choices are $D = 3, 4, 6$.

For the sake of simplicity, let us consider only the non compact case. In the case $D = 3,\ d = 2,\ \ \zeta(s|\mathcal{L})$ is regular at the origin and furthermore $\zeta(0|\mathcal{L}) = 0$. Thus the vacuum energy is given simply by

$$
\begin{aligned}
< E > &= \frac{1}{2}\zeta(-\frac{1}{2}|\mathcal{L}_Y) \\
&= \frac{\mathrm{Vol}(\mathbb{R}^2)a^{3/2}}{32\pi^{3/2}}\sum_{k=0}^{\infty}\frac{(-\tilde{M}^2)^k}{k!}\Gamma\left(\frac{2k-1}{4}\right)\Gamma\left(\frac{2k-3}{4}\right), \qquad (3.19)
\end{aligned}
$$

while for $D = 6$, $d = 2$ and $p = 4$, from Eq. (2.6) we have

$$
\begin{aligned}
< E > = -\frac{\mathrm{Vol}(\mathbb{R}^5)a^3}{256\pi^3}\Bigg[&\sum_{k=0,\ k\neq 1,3}^{\infty}\frac{(-\tilde{M}^2)^k}{k!}\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{k-3}{3}\right) \\
&-\tilde{M}^2\frac{\tilde{M}^4-6}{12}\left(1+2\pi\left(\log\frac{\mu^2}{2a}+\frac{1}{2\sqrt{\pi}}(\Psi(1)-\Psi(1/2)\right)\right)\Bigg]. \qquad (3.20)
\end{aligned}
$$

In Eqs. (3.19) and (3.20) $\tilde{M}^2 \equiv M^2/a$ and this factor is dimensionless and independent on the coupling constant.

The situation changes for $D = 6$ $(d = 4,\ \ p = 2)$ and $D = 4$ $(p = d = 2)$. In these cases the zeta function has a pole at the origin and one has to make use of Eq. (3.17). Let us first consider the case $D = 6,\ d = 4,\ p = 2$. We have

$$
\begin{aligned}
< E > = -\frac{\mathrm{Vol}(\mathbb{R}^5)a^3}{256\pi^3}\Bigg[&\sum_{k=0,\ k\neq 1,3}^{\infty}\frac{(-\tilde{M}^2)^k}{k!}\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{k-3}{2}\right) \\
&+\tilde{M}^2\left(\frac{1}{2}\Psi'(1)+\left(\log\frac{\mu^2}{a}-\gamma\right)^2+2+(4-\frac{\tilde{M}^4}{3})\left(\log\frac{\mu^2}{a}-\gamma\right)\right)\Bigg]. \qquad (3.21)
\end{aligned}
$$

In the other case, $D = 4$, $p = d = 2$, Eq. (3.17) gives

$$
\begin{aligned}
< E > = -\frac{\mathrm{Vol}(\mathbb{R}^3)a^2}{64\pi^2}\Bigg[&\sum_{k=1,\ k\neq 2}^{\infty}\frac{(-\tilde{M}^2)^k}{k!}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k-2}{2}\right)-2-\frac{4\pi^2}{3} \\
&+8\gamma^2-8\left(\frac{1}{2}-2\gamma+\log\frac{\mu^2}{a}\right)^2+\tilde{M}^4(\log\frac{\mu^2}{2a}+\frac{1}{2}\Psi(1)-\frac{1}{2}\Psi(1/2))\Bigg]. \qquad (3.22)
\end{aligned}
$$

Our evaluation of the functional determinant corresponding to the thermal fluctuation operator has also led to a non-analytic dependence on the coupling constant λ, corresponding to the resummation of all the one-loop diagrams. As is well known, higher-loop diagrams could, in principle, give a contribution of the same order. In any case, our computation is the necessary first step towards the final result.

In the compact case, we only obsserse that we have qualitatively the same results, but they are much more involved. Here, the sign of vacuum energy is relevant for issues concerning the stabilization mechanism of the compactification radius. However, it is to be noticed that its value depends on the renormalization parameter μ and on the number of non commuting coordinates.

3.3. **High temperature expansion.** To obtain the high temperature expansion, $\beta \to 0$, one may use Eqs. (2.14) and (3.15). We consider only the case $D = 4$ since for the other cases, D odd or $D = 6$, the high temperature expansion is standard, in its first leading terms.

For suitable s, we shift the vertical contour in the second term of Eq. (2.14) to the left. There are several simple poles and double poles in z. The simple pole at $z = 0$ gives a contribution which cancels with the first term on the r.h.s. of Eq. (2.14). Then there are simple poles at $z = 1$ and $z = 4 - 2r - 2s$. The double poles occur at $z = -4k - 2s$. As a result, the thermal zeta function reads

$$\begin{aligned}\zeta(s|\mathcal{L}) = \zeta(s|\mathcal{L}_Y) + \frac{\beta}{(4\pi)^{\frac{1}{2}}\Gamma(s)}\Bigg[\left(\frac{\beta}{2}\right)^{-4} A_0(\mathcal{L}_Y)\zeta_R(4)\\ + \left(\frac{\beta}{2}\right)^{-2} A_1(\mathcal{L}_Y)\zeta_R(2) + A_2(\mathcal{L}_Y)\zeta_R(-1) + \mathcal{O}(s)\Bigg]\\ -\frac{\beta^{2s+1}B_0(\mathcal{L}_Y)}{(2\pi)^{2s+1}\Gamma(s)}\Bigg[-\log\left(\frac{\beta}{2\pi}\right)\Gamma(s+1/2)\zeta_R(1+2s)\\ -\frac{1}{2}\frac{d}{ds}\Gamma(s+1/2)\zeta_R(1+2s) - \Gamma(s+1/2)\frac{d}{ds}\zeta_R(1+2s)\Bigg] + \frac{\mathcal{O}(\beta^5)}{\Gamma(s)}.\end{aligned} \tag{3.25}$$

It should be noted that the high temperature expansion is still singular at $s = 0$ due to the $(d/ds)\zeta_R(1 + 2s)$ term, and, strictly speaking, one should make use of Eq. (3.12) in order to compute the first quantum correction to the partition function. However, this term originates a subleading contribution, of order zero in β, the leading contributions being similar to those in the non-commutative case.

4. Conclusions

In this paper we have calculated the first quantum correction to the partition function at finite temperature for massless scalar fields on flat manifolds with non compact and with compact non-commutative dimensions. Dimensional regularization implemented with zeta-function techniques has been used to perform the calculations, which are rendered absolutely rigorous through the use of such machinery. This is not a trivial issue. In fact, the difficulties posed by the appearance of the non-commutative contributions in the spectral functions (which depend on it non analytically) should not be underestimated. Our calculation should be viewed as an attempt at an implementation of the background field method to the non-commutative case.

On the positive side, this has as a consequence that the quantum theory at finite temperature obtained by 'addition' of the non-commutative dimensions, departs appreciably from the commutative case. This is true concerning the mathematics of the problem: new poles in the zeta function, new asymptotic expansions, as well as the physics of the model. Here we have investigated the changes suffered by the vacuum energy density in the cases considered. After obtaining the general formula for the family of spacetimes at issue, a new prescription for the regularized vacuum energy has been obtained.

This new prescription has been exemplified with the explicit expressions corresponding to dimensions $D = 3, 4, 6$, where changes in the coefficients of the corresponding expressions of the regularized vacuum energy density are notorious.

One should also notice that, in the case of the presence of the pole at the origin for the zeta function, the dependence on the renormalization parameter in the vacuum energy (or in the effective action) is quite complicated, and, as a consequence, the one-loop renormalization group equations will be different from the standard ones. Furthermore, making use of the Eq. (A.10), contained in the Appendix of Ref. [36], it is possible to show that the use of a different regularization, for example the cut-off regularization, leads to the same conclusions.

We have also discussed the high temperature expansion for the partition function and have shown that new subleading terms, associated with the non-commutative part of the operator spectrum, are present. In subsequent work we aim at exploring the subject further, by trying to approach an experimental situation where one could actually check the non-commutative results vs the ordinary, commutative ones.

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